Relativistic and Classical Doppler Electronic Tracking Accuracies

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The first-order Doppler effect has, in general, been used in electronic tracking, but for the greater accuracies now desired in velocity measurements, this approach is no longer satisfactory. A fundamental study of the Doppler effect is presented in this paper. Five steps are developed which enable the exact derivation of the Doppler equations for any system. Six different configurations of transmitter, receiver, and vehicle are investigated, and the results are applied to a number of Doppler systems. It is determined that, for velocity inaccuracies ≤1.0 fps, the second-order relativistic or classical equations must be used. The receipt of a zero Doppler shift has also been investigated, and it does not necessarily imply zero line-of-sight velocity.

Nomenclature

a = proportional frequency shift factor produced by a beacon
b = constant frequency shift factor produced by a beacon
c = velocity of light
f = frequency
h = factor relating the constant frequency shift to the proportional shift produced by a beacon
$\mathbf{N}$ = wave normal of a plane wave
t = time parameter
v = velocity magnitude
$\mathbf{V}$ = velocity relative to the earth
w = phase velocity
$\mathbf{X}$ = position vector

Subscripts

i = receiver-fixed reference system when subscript 0 refers to transmitter
d = Doppler received by a beacon
r = receiver reference system
t = transmitter reference system
v = vehicle
0 = fixed reference system (receiver or transmitter as the case may be)

Introduction

IN the operation of a space or missile test program, one of the critical aspects is the receipt of trajectory information. This is true for both real-time control and for post-flight evaluation. Velocity is usually obtained by the Doppler effect, which is the shift in frequency of a transmitted signal caused by the relative motion of the vehicle and the station. The usual practice has been to use the first-order classical Doppler equations to relate the frequency shift to velocity. However, if velocity errors are to be reduced to a few tenths of a foot per second and less, this approach is no longer permissible. At least the second-order classical Doppler equation is required, and, for certain Doppler systems, the second-order relativistic Doppler equation must be used.

Because of the past sufficiency of the first-order terms in the equations, the unnecessary second-order terms usually contain any inaccuracies in the derivations. However, with the present need for greater accuracies in these second-order terms, a fresh look at the Doppler effect in electronic tracking is of considerable value. The purpose of the present study is to establish the fundamental steps needed for a correct derivation of the Doppler equation for any type of Doppler system, to derive the Doppler equations (classical and special relativistic) for a number of representative systems, and to determine the inaccuracies due to using the first-order equations so derived. With these equations on hand, several systems are evaluated, and the specified inaccuracies are compared with the inaccuracies inherent in the first-order equations. In addition, the phenomenon of zero Doppler shift is investigated, since this does not always imply zero line-of-sight velocity when the second-order equations are used.

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Fundamental Doppler Equations and Approximations

Derivation Philosophy

The key to the derivation of the Doppler effect for any configuration of transmitter, receiver, and beacon (or reflector) lies in the following sequence of steps.

1) Equate the phase of the radar wave as observed by the transmitter to that observed by the receiver, and so forth. The phase is an invariant and is identical in any coordinate system be it transmitter, receiver, or beacon.

2) Select the coordinate system to which all measurements will be referenced.

3) Establish the relative velocity parameters between sequential portions of the configuration, i.e., transmitter and beacon, beacon and receiver, etc. The Doppler effect depends only on relative velocity.

4) Apply the Galilean transformation (nonrelativistic) or the Lorentz transformation (relativistic) to the phase equalities.

5) Solve the resulting equations for the Doppler effect.

Consider the application of the previous steps to the configuration depicted in Fig. 1. The transmitter is fixed on earth. The receiver is fixed on a vehicle traveling with a velocity $\mathbf{V}$ relative to the earth. $\mathbf{N}$ is the wave normal of a plane wave, which is described in the transmitter reference system by one or several wave functions of the form $\cos 2\pi f \times [t - \mathbf{N}\cdot\mathbf{X}/w]$.

For step 1, the phase is equated

$$f[t - \mathbf{N}\cdot\mathbf{X}/w] = f'[t' - \mathbf{N}'\cdot\mathbf{X}'/w'] \quad (1)$$

where the primed parameters are the measurements in the receiver's reference system. For step 2, the earth-based transmitter reference system is chosen. This means that the

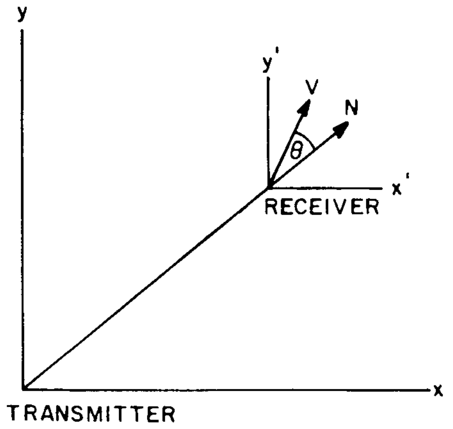

Fig. 1 Representative configuration.

final equation relating f and f' will contain no primed parameters. For step 3, the relative velocity for this simple configuration is just **V**.

For step 4, the inverse Lorentz transformation [see Appendix, Eq. (A4)] is applied. Thus,

$$f\{\gamma[t' + \mathbf{V}\cdot\mathbf{X}'/c^2] - \mathbf{N}\cdot\mathbf{X}'/w - \mathbf{N}\cdot\mathbf{V}[(\gamma - 1)\mathbf{X}'\cdot\mathbf{V}/v^2 + \gamma t']/w\} = f'\{t' - \mathbf{N}'\cdot\mathbf{X}'/w'\} \quad (2)$$

where $\gamma = (1 - \beta^2)^{-1/2}$ and $\beta = v/c$. Equation (2) must be satisfied for all values of the independent variables t', $\mathbf{X}'$. This is possible only if the coefficients of t', $\mathbf{X}'$, respectively, on both sides are equal. Equating the coefficients of t' gives

$$f[\gamma - \mathbf{N}\cdot\mathbf{V}\gamma/w]t' = f'[1]t' \quad (3)$$

Therefore, the Doppler frequency f' observed by the receiver on the vehicle is

$$f' = f(1 - \mathbf{N}\cdot\mathbf{V}/w)(1 - \beta^2)^{-1/2} \quad (4)$$

where $\mathbf{N}\cdot\mathbf{V}$ is the scalar product of the two vectors, and **N** is a unit vector. From Fig. 1, $\mathbf{N}\cdot\mathbf{V} = v\cos\theta$. Some pitfalls that occur in deriving Doppler equations are discussed in the Appendix. Equation (4) is the relativistic Doppler effect. Allowing $\beta \to 0$ results in the classical Doppler effect, which would of course result from using the Galilean transformation. It is interesting to note that equating the coefficients of the vector components of the $\mathbf{X}'$ variable on both sides of Eq. (2) would result in the determination of the remaining primed parameters as functions of the unprimed ones.

Specific Configurations

In configuration A (Fig. 2), the transmitter is stationary and the (vehicle) receiver is in motion. This configuration is identical to that discussed previously where

$$f_t = f \qquad f_r = f' \qquad \mathbf{V} = \mathbf{V} \quad (5)$$

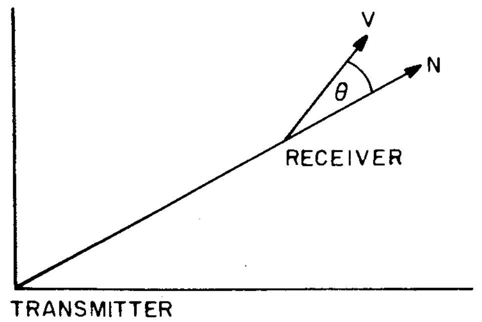

Fig. 2 Configuration A, earth-fixed reference.

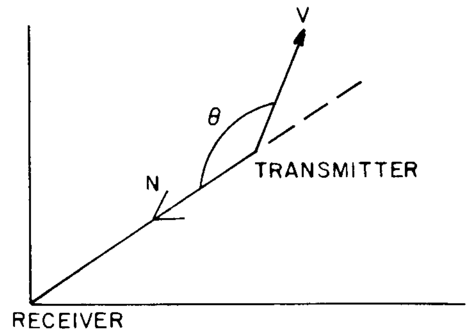

Fig. 3 Configuration B, earth-fixed reference.

Therefore, from Eq. (4), the relativistic Doppler frequency is

$$f_r = f_t(1 - \mathbf{N}\cdot\mathbf{V}/w)(1 - \beta^2)^{-1/2} \quad (5a)$$

where $\mathbf{N}\cdot\mathbf{V} = v\cos\theta$. Allowing $\beta \to 0$ results in the classical Doppler frequency

$$f_r = f_t[1 - \mathbf{N}\cdot(\mathbf{V} + \Delta\mathbf{V})/w] \quad (5b)$$

where $(\mathbf{V} + \Delta\mathbf{V})$ is the inaccurate velocity obtained by using Eq. (5b) instead of (5a). Equation (5) contains earth-fixed reference parameters. Expanding (5a), and retaining up to second-order terms only, results in

$$f_r \approx f_t[1 - \mathbf{N}\cdot\mathbf{V}/w + \beta^2/2] \quad (6)$$

Subtracting Eq. (5b) from (6) results in

$$\mathbf{N}\cdot\Delta\mathbf{V} = \Delta v\cos\theta \approx -v^2/2c \quad (7)$$

where it has been assumed that $w/c \approx 1$. Equation (7) is the inaccuracy in the velocity determination using the classical Doppler equation.

In configuration B (Fig. 3), the (vehicle) transmitter is in motion and the receiver is stationary. The vehicle's reference system is still considered the primed system as in configuration A. Therefore, $f_t = f'$, $f_r = f$, $\mathbf{V} = \mathbf{V}$, and, from Eq. (4),

$$f_r = \frac{f_t(1 - \beta^2)^{1/2}}{1 - \mathbf{N}\cdot\mathbf{V}/w} \quad (8a)$$

Again allowing $\beta \to 0$ results in

$$f_r = \frac{f_t}{1 - \mathbf{N}\cdot(\mathbf{V} + \Delta\mathbf{V})/w} \quad (8b)$$

Equation (8) also contains earth-fixed reference parameters. Expanding Eq. (8) results in

$$f_r \approx f_t[1 + \mathbf{N}\cdot\mathbf{V}/w + (\mathbf{N}\cdot\mathbf{V}/w)^2 - \beta^2/2] \quad (9a)$$

and, for $\beta \to 0$,

$$f_r \approx f_t\{1 + \mathbf{N}\cdot(\mathbf{V} + \Delta\mathbf{V})/w + [\mathbf{N}\cdot(\mathbf{V} + \Delta\mathbf{V})/w]^2\} \quad (9b)$$

Subtracting Eq. (9b) from (9a) results in

$$\mathbf{N}\cdot\Delta\mathbf{V} = \Delta v\cos\theta \approx -\frac{v^2/2c}{1 + 2\mathbf{N}\cdot\mathbf{V}/c} \quad (10a)$$

where it has been assumed that $w/c \approx 1$ and where $(\mathbf{N}\cdot\Delta\mathbf{V}/w)^2$ has been neglected. Comparing Eq. (10a) with (7) shows that the inaccuracy of configuration B is approximately that of A since $2\mathbf{N}\cdot\mathbf{V}/c \ll 1$ in Eq. (10a).

The usual practice is to neglect all second-order terms. It is of interest to note the resulting inaccuracy. Neglecting the second-order terms in Eq. (9a) results in an inaccuracy ΔV_1 given by

$$\mathbf{N}\cdot\Delta\mathbf{V}_1 = \Delta v_1\cos\theta \approx [-v^2/2c][1 - 2\cos^2\theta] \quad (10b)$$

where it has been assumed that $w/c \approx 1$. Since $-1 \leq [1 - 2\times$

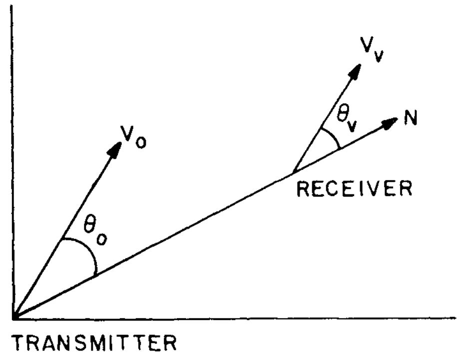

Fig. 4 Configuration *C*, case 1: transmitter-fixed reference.

$\cos^2\theta] \leq 1$, the procedure of neglecting second-order terms reduces the inaccuracy associated with the use of the classical Doppler equation. In fact, when $\theta = \pi/4, 3\pi/4$ there is effectively no inaccuracy [as Eq. (10b) indicates], but the sign of the inaccuracy may change.

In configuration C, both transmitter and receiver are in motion. Two cases are considered: the tracked vehicle first containing the receiver (Fig. 4) and the second containing the transmitter (Fig. 5).

In case 1, the receiver in the tracked vehicle is identical to configuration A with

$$\left.\begin{aligned} \mathbf{V} = \mathbf{V}_v - \mathbf{V}_0 \qquad & \Delta\mathbf{V} = \mathbf{V}_v \\ \mathbf{N}\cdot\mathbf{V} = \mathbf{N}\cdot(\mathbf{V}_v - \mathbf{V}_0) &= v_v \cos\theta_v - v_0\cos\theta_0 \\ \mathbf{N}\cdot\Delta\mathbf{V} = \mathbf{N}\cdot\Delta\mathbf{V}_v &= \Delta v_v \cos_v \\ v^2 = v_v^2 + v_0^2 - 2\mathbf{V}_v\cdot\mathbf{V}_0 \qquad & \beta = v/c \end{aligned}\right\} \tag{11}$$

Equations (5–7) apply directly with the modifications given by Eqs. (11). These adapted equations now contain transmitter-fixed reference parameters.

It is of interest to consider the situation where $\mathbf{V}_0$ is antiparallel to $\mathbf{V}_v$, $\theta_0 = 180 - \theta_v$. From Eq. (7), modified by (11), the inaccuracy using the classical Doppler equation is

$$\mathbf{N}\cdot\Delta\mathbf{V}_v = \Delta v_v \cos\theta_v \approx [-v_v^2/2c][1 + 2(v_0/v_v) + (v_0/v_v)^2] \tag{12}$$

For a satellite tracking station, traveling antiparallel to the tracked vehicle at approximately the same velocity, the inaccuracy would be four times that of an earth-based tracking station, as indicated by Eq. (12).

In case 2, the transmitter in the tracked vehicle, is identical to configuration B with the modifications given by Eqs. (11). Equations (8–10), modified by (11), apply directly. These adapted equations now contain receiver-fixed reference parameters.

Consider again $\mathbf{V}_0$ antiparallel to $\mathbf{V}_v$, $\theta_0 = 180 - \theta_v$. From Eq. (10a), modified by (11), the inaccuracy using the classical Doppler equation is

$$\mathbf{N}\cdot\Delta\mathbf{V}_v = \Delta v_v \cos\theta_v \approx \frac{[-v_v^2/2c][1 + 2v_0/v_v + (v_0/v_v)^2]}{1 + 2\mathbf{N}\cdot\mathbf{V}/c} \tag{13}$$

Since $2\mathbf{N}\cdot\mathbf{V}/c \ll 1$, the discussion in case 1 applies here as well.

In configuration D (Fig. 6), the transmitter is stationary, the receiver is stationary, and the vehicle is in motion. In the general case, the vehicle receives the Doppler frequency f_d, and the beacon retransmits $(af_d + b)$ to the receiver. For the special case of skin tracking, $a = 1$ and $b = 0$.

Consider the Doppler frequency f_0 received by the vehicle. This portion is identical to configuration A with f_d substituted for f_r in Eq. (5a). Therefore,

$$f_d = f_t(1 - \mathbf{N}_t\cdot\mathbf{V}/w)(1 - \beta^2)^{-1/2}$$

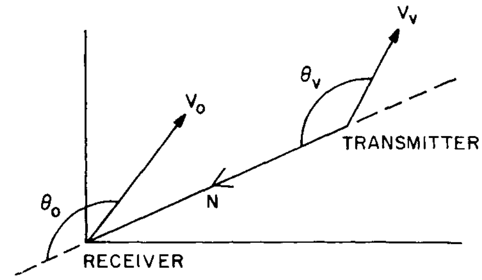

Fig. 5 Configuration *C*, case 2: receiver-fixed reference.

The retransmitted (or reflected) frequency is $(af_d + b)$. The Doppler frequency received by the receiver is identical in form to configuration B with $(af_d + b)$ substituted for f_t in Eq. (8a). Therefore,

$$f_r = (af_d + b)(1 - \beta^2)^{1/2}/1 - \mathbf{N}_r\cdot\mathbf{V}/w$$

Eliminating f_d from the two preceding equations results in

$$f_r = af_t\,\frac{(1 - \mathbf{N}_t\cdot\mathbf{V}/w) + h(1 - \beta^2)^{1/2}}{1 - \mathbf{N}_r\cdot\mathbf{V}/w} \tag{14a}$$

where $h = b/af_t$. Allowing $\beta \to 0$ results in the classical Doppler equation

$$f_r = af_t\,\frac{1 - \mathbf{N}_t\cdot(\mathbf{V} + \Delta\mathbf{V})/w + h}{1 - \mathbf{N}_r\cdot(\mathbf{V} + \Delta\mathbf{V})/w} \tag{14b}$$

which contains earth-fixed reference parameters.

Consider the skin-tracking case where $a = 1$ and $b = 0$. Comparison of Eqs. (14a) and (14b) shows that the relativistic and classical Doppler equations are identical, and $\Delta V = 0$. This is also true for beacon tracking where $b = 0$ and a has any value.

Expanding Eq. (14) through second-order terms yields

$$f_r = af_t\{(1 + h) + (1 + \mathbf{N}_r\cdot\mathbf{V}/w)[(1 + h)\mathbf{N}_r\cdot\mathbf{V}/w - \mathbf{N}_t\cdot\mathbf{V}/w] - \beta^2 h/2\} \tag{15a}$$

and, for $\beta \to 0$,

$$f_r = af_t\{(1 + h) + [1 + \mathbf{N}_r\cdot(\mathbf{V} + \Delta v)/w]\cdot[(1 + h)\mathbf{N}_r\cdot(\mathbf{V} + \Delta\mathbf{V})/w - \mathbf{N}_t\cdot(\mathbf{V} + \Delta\mathbf{V})/w]\} \tag{15b}$$

Subtracting Eq. (15b) from (14a) results in

$$(\mathbf{N}_t - \mathbf{N}_r)\cdot\Delta\mathbf{V} = \Delta v(\cos\theta_t - \cos\theta_r) \approx \frac{(v^2h/2c)(\cos\theta_t - \cos\theta_r)}{(1 + 2N_r\cdot\mathbf{V}/w)[-\cos\theta_t - (h + 1)\cos\theta_r]} \tag{16a}$$

where $\mathbf{N}_r\cdot\Delta\mathbf{V}\mathbf{N}_t\cdot\Delta\mathbf{V}/w^2$ and $(1 + h)\ (\mathbf{N}_r\cdot\Delta\mathbf{V}/w)^2$ have been

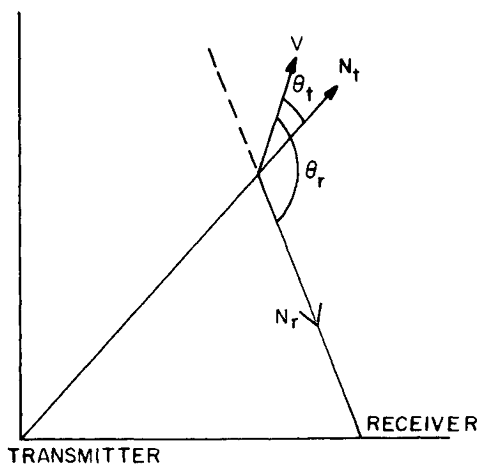

Fig. 6 Configuration *D*, earth-fixed reference.

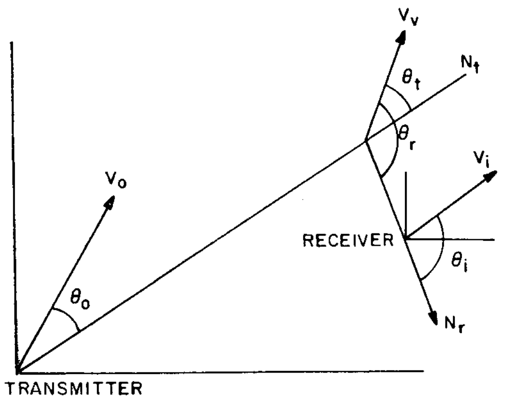

Fig. 7 Configuration *E*, transmitter-fixed reference and receiver-fixed reference.

neglected, and it has been assumed that $w \approx c$. Equation (16a) is the inaccuracy using the classical Doppler equation for beacon tracking where $b \neq 0$ ($h \neq 0$). Let us examine the inaccuracy in the usual practice of dropping the second-order terms for skin tracking or beacon tracking ($b = 0$, $h = 0$). Letting $h = 0$ in Eq. (15a) results, of course, in the identical relativistic and classical Doppler equation as discussed previously. Dropping second-order terms and solving for the inaccuracy $\Delta\mathbf{V}_1$ yields

$$(\mathbf{N}_t - \mathbf{N}_r)\cdot\Delta\mathbf{V}_1 = \Delta v_1(\cos\theta_t - \cos\theta_r) \approx (v^2/w)[\cos\theta_t\cos\theta_r - \cos^2\theta_r] \quad (16b)$$

Solving for Δv_1 gives $\Delta v_1 \approx v^2\cos\theta_r/w$, which is independent of the transmitter angle θ_t.

In configuration *E* (Fig. 7), the transmitter, receiver, and vehicle are in motion. Consider the case of skin tracking or beacon tracking with no beacon frequency shifts ($a = 1$, $b = 0$; $h = 0$ of configuration *D*). The derivation is identical to that of configuration *D* with the modifications as indicated.

Eliminating f_d from the equations for the Doppler frequencies, seen by the vehicle and the receiver, results in

$$f_r = f_t\frac{1 - \mathbf{N}_t\cdot\mathbf{V}_t/w_t\;(1-\beta_r^2)^{1/2}}{1 - \mathbf{N}_r\mathbf{V}_r/w_r\;(1-\beta_t^2)^{1/2}} \quad (17a)$$

where

$$v_t = \mathbf{V}_v - \mathbf{V}_0 \qquad \mathbf{V}_r = \mathbf{V}_v - \mathbf{V}_i$$
$$\mathbf{N}_t\cdot\mathbf{V}_t = v_v\cos\theta_t - v_0\cos\theta_0$$
$$\mathbf{N}_r\cdot\mathbf{V}_r = v_v\cos\theta_r - v_i\cos\theta_i$$
$$v_t^2 = v_v^2 + v_0^2 - 2\mathbf{V}_v\cdot\mathbf{V}_0$$
$$v_r^2 = v_v^2 + v_i^2 - 2\mathbf{V}_v\cdot\mathbf{V}_i$$

Allowing $\beta_r, \beta_t \to 0$ results in the classical Doppler equation

$$f_r = f_t\frac{1 - \mathbf{N}_t\cdot(\mathbf{V}+\Delta\mathbf{V})/w_t}{1 - \mathbf{N}_r\cdot(\mathbf{V}+\Delta\mathbf{V})/w_r} \quad (17b)$$

where $\Delta\mathbf{V} = \Delta\mathbf{V}_v$. If the transmitter and receiver have the same velocity vector ($\mathbf{V}_0 = \mathbf{V}_i$), then the radicals cancel in Eq. (17a) and the classical and relativistic equations are identical.

Expanding Eq. (17a) yields

$$f \approx f_t\{1 + (1 + \mathbf{N}_r\cdot\mathbf{V}_r/w_r)[\mathbf{N}_r\cdot\mathbf{V}_r/w_r - \mathbf{N}_t\cdot\mathbf{V}_t/w_t] + (\beta_t^2 - \beta_r^2)/2\} \quad (18a)$$

where

$$(\beta_t^2 - \beta_r^2)/2 = (v_0^2 - v_i^2)/2c^2 + \mathbf{V}_v/c\cdot(\mathbf{V}_i - \mathbf{V}_0)/c = (\mathbf{V}_0 - \mathbf{V}_i)\cdot[(\mathbf{V}_0 + \mathbf{V}_i)/2 - \mathbf{V}_v]/c^2$$

and, for $\beta_t, \beta_r \to 0$,

$$f_r \approx f_t\{1 + [1 + \mathbf{N}_r\cdot(\mathbf{V}_r + \Delta\mathbf{V})/w_r] \times [\mathbf{N}_r\cdot(\mathbf{V}_r+\Delta\mathbf{V})/w_r - \mathbf{N}_t\cdot(\mathbf{V}_t+\Delta\mathbf{V})/w_t]\} \quad (18b)$$

Subtracting Eq. (18b) from (18a) yields the inaccuracy

$$(\mathbf{N}_t - \mathbf{N}_r)\cdot\Delta\mathbf{V} = \Delta v(\cos\theta_t - \cos\theta_r) \approx -\frac{[(\mathbf{V}_0 - \mathbf{V}_i)/c]\cdot[\mathbf{V}_v - (\mathbf{V}_0 + \mathbf{V}_i)/2]}{(1 + \mathbf{N}_r\cdot\mathbf{V}_r/c) - [\mathbf{N}_t\cdot\mathbf{V}_t/c - \mathbf{N}_r\mathbf{V}_r/c]/[1 - \cos\theta_t/\cos\theta_r]} \quad (19a)$$

where it has been assumed that $w_t \approx w_r \approx c$, and where $\mathbf{N}_r\cdot\Delta\mathbf{V}\mathbf{N}_t\cdot\Delta\mathbf{V}/w^2$ and $(\mathbf{N}_r\cdot\Delta\mathbf{V}/w)^2$ have been neglected. Equation (19a) is the inaccuracy using the classical Doppler equation. This inaccuracy is zero when $\mathbf{V}_0 = \mathbf{V}_i$ and when $\mathbf{V}_v = (\mathbf{V}_0 + \mathbf{V}_i)/2$.

Let us examine the case where $\mathbf{V}_0 = \mathbf{V}_i$, ($\mathbf{V}_r = \mathbf{V}_t$). From Eq. (18a), with $w_t \approx w_r = w$,

$$f_r \approx f_t\{1 + (1 + \mathbf{N}_r\cdot\mathbf{V}_t/w)[\mathbf{N}_r\cdot\mathbf{V}_t/w - \mathbf{N}_t\cdot\mathbf{V}_t/w]\}$$

which is identical to Eq. (15a) of configuration *D* with $h = 0$ and $\mathbf{V}$ replaced by $\mathbf{V}_t$.

The inaccuracy $\Delta\mathbf{V}_i$, in dropping second-order terms, is therefore given by Eq. (16b) with v replaced by v_t or v_r since for this case $v_t = v_r$. Therefore

$$\Delta v_1 \approx v_r^2\cos\theta_r/w = [v_v^2/w][1 + (v_i/v_v)^2 - 2V_v\cdot V_i/v_v^2] \times \cos\theta_r \quad (19b)$$

If the vehicle is traveling antiparallel to the receiver (also transmitter), then

$$\Delta v_1 \approx [v_v^2/w][1 + 2(v_i/v_v) + (v_i/v_v)^2]\cos\theta_r$$

For $v_i = v_v$, the inaccuracy is four times that of the similar earth-fixed system given by Eq. (16b) in configuration *D*.

Zero Doppler Shift Effect

In the use of the Doppler systems, it is generally assumed that the receipt of a zero Doppler shift implies that the line-of-sight velocity of the vehicle is identically zero. This is, of course, inaccurate, and the degree of inaccuracy is dependent upon the particular configuration in use. The degree of inaccuracy for configuration *B* is analyzed below, and the inaccuracies associated with other configurations are briefly indicated. For completeness, it should perhaps be stated that there is a zero Doppler shift for the trivial case of zero relative motion for all configurations.

From Eq. (8a), the zero Doppler shift occurs in configuration *B* when

$$\mathbf{N}\cdot\mathbf{V}/w = 1 - (1 - \beta^2)^{1/2} \quad (20a)$$

From an examination of Fig. 3 in conjunction with Eq. (20a), it is clear that the zero Doppler shift occurs only when the vehicle is approaching the earth-fixed receiver. Expanding Eq. (20a), and letting $w \approx c$, yields

$$\mathbf{N}\cdot\mathbf{V} = v\cos\theta \approx v^2/2c \quad (20b)$$

which is the line-of-sight velocity inaccuracy when the zero Doppler shift is implied to mean zero line-of-sight velocity.

For configuration *A*, the zero Doppler shift is identical to that of configuration *B* except that the effect occurs when the vehicle is receding from the transmitter. For configuration *C*, case 1 is identical to configuration *A*; case 2 is identical to configuration *B*. For configuration *D*, setting Eq. (14a) equal to ($af_t + b$), expanding, and letting $w \approx c$, results in

$$[\mathbf{N}_t - (1 + h)\mathbf{N}_r]\cdot\mathbf{V} \approx -hv^2/2c \quad (21a)$$

For the case of skin tracking (or beacon with $b = 0$), $h = 0$, and the zero shift effect occurs when

$$\mathbf{N}_t\cdot\mathbf{V} = \mathbf{N}_r\cdot\mathbf{V} \quad (21b)$$

The Doppler shift produced by the vehicle receding from the transmitter is exactly cancelled by that produced by the vehicle approaching the receiver. For collocation of receiver and transmitter ($\mathbf{N}_t = -\mathbf{N}_r$), the zero shift occurs only when the line-of-sight velocity is identically zero. For the general case exhibited by Eq. (21b), the inaccuracy is thus $\mathbf{N}_t \cdot \mathbf{V}$ (or $\mathbf{N}_r \cdot \mathbf{V}$).

For configuration E, setting Eq. (17a) equal to f_t, expanding, and letting $w_t \approx w_r \approx c$, yields

$$\mathbf{N}_t \cdot \mathbf{V}_t - \mathbf{N}_r \cdot \mathbf{V}_r \approx (v_t^2 - v_r^2)/2c \quad (22a)$$

For the case where $\mathbf{V}_0 = \mathbf{V}_i$,

$$(\mathbf{N}_t - \mathbf{N}_r) \cdot \mathbf{V}_v = (\mathbf{N}_t - \mathbf{N}_r) \cdot \mathbf{V}_0 \quad (22b)$$

Actual Doppler Systems: Relativistic and Classical Approximation Inaccuracies

The Doppler equations previously derived are now used to examine a number of Doppler systems. The examination is limited to a comparison of the system accuracy requirements with the inaccuracies resulting from either the use of classical equations (in place of relativistic) or from the practice of neglecting second-order terms in the equations. In addition, the effect of receiving a zero Doppler shift is investigated.

For purposes of numerical computations, the following values are used: velocity of light, $c \approx 1.0 \times 10^9$ fps; escape velocity, $v \approx 3.5 \times 10^4$ fps; and orbital velocity, $v \approx 2.5 \times 10^4$ fps.

GLOTRAC: Global Tracking Network

GLOTRAC[1] is being designed and constructed to meet the tracking requirements of advanced satellite and space-probe programs. GLOTRAC contains a number of measurement subsystems and techniques. The Doppler principle is used in range-rate measurements. A minimum of three range-rate stations operate simultaneously on the signals radiated by the transponder. One of the three stations sends either 5060.194 mc (type C transponder) or 5052.0833 mc (type G transponder) signals to the transponder as interrogation signals. Range-rate data obtained at a station equipped with a transmitter and receiver will have an accuracy of 0.09 fps. Stations equipped only with receivers will have an accuracy of 0.51 fps.

The GLOTRAC Doppler subsystem is a configuration D system. Operation of the type C transponder is as follows: 1) transmitter interrogation frequency f_t = 5060.194 mc; 2) beacon receives Doppler frequency f_d; 3) beacon retransmits the frequency of $af_d + b$ where $a = 1$, $b = -60.194$ mc; 4) receivers receive Doppler frequency f_r. For these values, $h = b/af_t \approx -0.01$. From Eq. (16a), for h and $\mathbf{N}_r \cdot \mathbf{V}/c \ll 1$, the inaccuracy resulting from the use of the classical Doppler is

$$\Delta v \ (\cos\theta_t - \cos\theta_r) \approx v^2 h/2c \quad (23a)$$

For the master station (receiver and transmitter) $\cos\theta_r = -\cos\theta_t$, and Eq. (23a) becomes

$$\Delta v \cos\theta_t \approx v^2 h/4c \quad (23b)$$

For the remote stations (receiver only), for the case where $\theta_t \approx \theta_r$, Eq. (16a) yields

$$\Delta v \cos\theta_r \approx -v^2/2c \quad (23c)$$

which is independent of the h parameter. From Eq. (21a), for the master station, a zero Doppler shift is received when

$$\Delta v \cos\theta_t \approx -v^2 h/4c \quad (24a)$$

For the remote stations, for the case when $\theta_t \approx \theta_r$, Eq. (21a) yields

$$\Delta v \cos\theta_r \approx v^2/2c \quad (24b)$$

which is independent of the h parameter.

Table 1 GLOTRAC (transponder C) inaccuracies

Type	Eq.	Inaccuracies, fps: Escape velocity	Orbital velocity
For master station (specified accuracy, 0.09 fps)			
Classical	(23b)	−0.003	−0.0015
Zero-shift	(24a)	0.003	0.0015
For remote station (specified accuracy, 0.51 fps)			
Classical	(23c)	−0.6	−0.3
Zero-shift	(24b)	0.6	0.3

The numerical results are listed in Table 1. The numerical values, given at the beginning of this section, have been employed. The specified accuracies include uncorrectable propagation errors and atomic frequency standard offset errors. From Table 1, it appears that the GLOTRAC (transponder C) range-rate system should utilize the relativistic rather than the classical Doppler equations at the remote stations. In addition, allowance should be made for the fact that the receipt of a zero Doppler frequency does not necessarily imply zero line-of-sight velocity.

Operation of the type G transponder is as follows: 1) transmitter interrogation f_t = 5052.0833 mc; 2) beacon receives Doppler frequency f_d; 3) beacon retransmits the frequency $af_d + b$ where $a = 96/97$, $b = 0$; 4) receivers receive Doppler frequency f_r. The parameter h is therefore $h = b/af_t = 0$.

Comparison of Eqs. (14a) and (14b) shows that the relativistic and classical Doppler equations are identical. Therefore, there is no inaccuracy using the classical equation and $\Delta v = 0$. However, there is an inaccuracy Δv_1 that results from the practice of neglecting second-order terms in the classical expansion, which is given by Eq. (15b) with $h = 0$. From Eq. (16b), this inaccuracy is

$$\Delta v_1 \approx (v^2/c) \cos\theta_r \quad (25a)$$

For the master station, $\cos\theta_r = -\cos\theta_t$ and Eq. (25a) becomes

$$\Delta v_1 \approx -(v^2/c) \cos\theta_t \quad (25b)$$

or the line-of-sight inaccuracy is

$$\Delta v_1 \cos\theta_t \approx -(v^2/c) \cos^2\theta_t \quad (25c)$$

For the remote stations, from Eq. (25a), the line-of-sight inaccuracy is

$$\Delta v_1 \cos\theta_r \approx (v^2/c) \cos^2\theta_r \quad (25d)$$

From Eq. (21b), for the master station, a zero Doppler shift is received when

$$\theta_r = \theta_t = \pi/2 \quad (26a)$$

and the line-of-sight velocity is identically zero. For the remote stations, from Eq. (21b), a zero Doppler shift is received when

$$v \cos\theta_t = v \cos\theta_r \quad (26b)$$

and the zero Doppler received implies a line-of-sight velocity equal to $v \cos\theta_t$ [or $v \cos\theta_r$ since $\theta_r = \theta_t$ from Eq. (26b)].

The numerical results are listed in Table 2 and follow the same procedure as Table 1. The approximation inaccuracies have been calculated for $\theta = 0$. From Table 2, it is clear that in using the classical equations the second-order terms would be retained to achieve the specified accuracies.

ANNA: Army, Navy, NASA, Air Force Satellite

ANNA[2] is a geodetic satellite that is expected to obtain fine measurements concerning the shape of the earth and to relate major geodetic data to each other and to the earth's center of mass. Specifically, the satellite experiments will

Table 2 GLOTRAC (transponder G) inaccuracies

Types	Eq.	Inaccuracies, fps	
		Escape velocity	Orbital velocity
For master station (specified accuracy, 0.09 fps)			
Approximation	(25c)	-1.2 fps	-0.6 fps
Zero-Shift	(26a)	0	0
For remote station (specified accuracy, 0.51 fps)			
Approximation	(25d)	1.2 fps	0.6 fps
Zero-Shift	(26b)	$\theta_r = \theta_t$	$\theta_r = \theta_t$

produce precision measurements related to angle, range, and range rate.

Range-rate information will be obtained by observation of the Doppler shift of ultra-stable transmissions from the satellite, and four frequencies will be broadcast continuously for this purpose. Frequencies for geodetic measurements will be 162–324 mc with a 54–216 mc pair reserved for refraction studies and as a backup in event of failure of prime tracking frequencies. All four frequencies will be coherent, so that tracking can be accomplished using any two. Transmitters will have lower power drain, and hence will be left on continuously to be available to observers throughout the world.

The ANNA Doppler subsystem is a configuration B system. From Eq. (10a) for $2\mathbf{N}\cdot\mathbf{V}/c \ll 1$, the inaccuracy resulting from the use of the classical Doppler equation is

$$\Delta v \cos\theta \approx -v^2/2c \approx -0.3 \text{ fps} \tag{27}$$

for orbital velocity, which is then the inaccuracy in the line-of-sight velocity when the relativistic equations are not used. However, if the classical Doppler is used, and the second-order terms neglected, this inaccuracy tends to reduce the inaccuracy produced by not using the relativistic Doppler. From Eq. (10b) the total inaccuracy is

$$\Delta v_1 \cos\theta \approx [-v^2/2c][1 - 2\cos^2\theta] \tag{28a}$$

Comparison with Eq. (27) shows that

$$0 \leq |\Delta v_1 \cos\theta| \leq 0.3 \text{ fps} \tag{28b}$$

and when $\theta = \pi/4, 3\pi/4$ there is no inaccuracy. However, from Eq. (28a) it is seen that the sign of the inaccuracy is changed.

From Eq. (20c), a zero Doppler shift is received when

$$v \cos\theta \approx v^2/2c \approx 0.3 \text{ fps} \tag{29}$$

Therefore, the receipt of a zero Doppler shift implies a line-of-sight velocity of 0.3 fps and not zero.

MISTRAM: Missile Trajectory Measurement System

MISTRAM[3] is a precision missile trajectory measurement system that will operate independently of other range systems at Air Force Eastern Test Range (AFETR) to acquire a launched vehicle, track its flight through space, and accurately measure its position and velocity vectors. The range-rate data are obtained by Doppler techniques at X band with a specified accuracy of 0.02 fps. The following steps apply:

1) The transmitter at the master or central station generates two CW-X-band frequencies, nominally 8148 mc and 7884 to 7892 mc. The higher frequency (the range signal) is very stable, whereas the lower frequency (the calibrated signal) is swept periodically over the indicated range. Therefore $f_t = 8148$ mc.

2) The airborne transponder receives the signals, amplifies frequency shifts by 68 mc, and retransmits back to earth. Therefore, the received Doppler frequency (the range signal) is f_d and the retransmitted frequency is $af_d + b$.

3) If the 68 mc is a proportional shift, then

$$a = \tfrac{8216}{8148} = \tfrac{2054}{2037} \qquad b = 0 \qquad h = 0$$

4) If the 68 mc is a constant shift, then $a = 1$ and $b = 68$ mc. The parameter h is therefore $h = b/af_t \approx 0.01$.

5) Receivers receive Doppler frequency f_r.

The MISTRAM Doppler subsystem is a configuration D system. The analysis is therefore similar to that of GLOTRAC. The results tabulated there also apply here. For the constant shift (see item 4) Table 1 applies with the specified accuracies changed to 0.02 fps. The GLOTRAC h was negative, whereas the MISTRAM h is positive. Therefore, the tabulated results (Table 1) for the master station are positive for MISTRAM. For the proportional shift (see item 3) Table 2 applies, again with the specified accuracies changed to 0.02 fps. Therefore, refer to the GLOTRAC analysis.

Summary and Conclusions

This study has taken a fundamental look at the Doppler effect in electronic tracking. Five steps have been developed which enable the derivation of the Doppler equations for any Doppler system. These steps have then been used to derive six different configurations of transmitter, receiver, and vehicle. It has been determined that for velocity inaccuracies ≤ 1 fps, the second-order relativistic equations must be used. In some Doppler systems, the relativistic and classical equations are identical. In those cases, the second-order equations must also be used.

The effect of receiving a zero Doppler shift has also been investigated. This effect does not imply zero line-of-sight velocity; allowance for this fact must be made when the Doppler information is transformed into velocity information.

GLOTRAC and MISTRAM Doppler systems have been examined. The evaluation clearly indicates that, to obtain the desired small inaccuracies, the second-order equations must be utilized.

Finally, in the Appendix there is a discussion of some pitfalls that occur when the equations are not carefully derived from the fundamental Doppler effect. The approximate first-order equations for different Doppler systems are identical in appearance, but the second-order equations do differ when correctly derived.

Appendix

Galilean and Lorentz Transformations

Consider two arbitrary systems of inertia described by Cartesian coordinates $\mathbf{X} = (x, y, z)$ and $\mathbf{X}' = (x', y', z')$ respectively. According to the Newtonian conceptions of space and time, the connection between the time parameters and the coordinate vectors $\mathbf{X}$ and $\mathbf{X}'$ for the same space point in the two coordinate systems is given by

$$\mathbf{X}' = \mathbf{X} - \mathbf{V}t \qquad t' = t \tag{A1}$$

where $\mathbf{V}$ is a vector denoting velocity and direction of motion of the primed system; t is the time and, for simplicity, it is assumed that the origin of the two systems coincide at the time $t = 0$. Thus, the time is considered an absolute quantity. Equation (A1) is oftened referred to as the "Galilean transformation."

Since the two systems of coordinates are completely equivalent, and since the unprimed system moves with the velocity $-\mathbf{V}$ relative to the primed system, the inverse transformation to Eq. (A1) is simply obtained by interchanging the primed and the unprimed variable and simultaneously replacing $\mathbf{V}$ by $-\mathbf{V}$:

$$\mathbf{X} = \mathbf{X}' + \mathbf{V}t' \qquad t = t' \tag{A2}$$

According to the special theory of relativity[4] this same connection is given by

$$\mathbf{X}' = \mathbf{X} + \mathbf{V}[(\gamma - 1)\mathbf{X}\cdot\mathbf{V}/v^2 - \gamma t] \tag{A3a}$$

$$t' = \gamma[t - \mathbf{V}\cdot\mathbf{X}/c^2] \tag{A3b}$$

where $\gamma = 1/(1 - \beta^2)^{1/2}$ and $\beta = v/c$. Equation (A3) is usually called the Lorentz transformation. The inverse transformation is again obtained by interchanging $(\mathbf{X}', t')$ and $(\mathbf{X}, t)$ and replacing $\mathbf{V}$ by $-\mathbf{V}$:

$$\mathbf{X} = \mathbf{X}' + \mathbf{V}[(\gamma - 1)\mathbf{X}'\cdot\mathbf{V}/v^2 + \gamma t'] \quad \text{(A4a)}$$

$$t = \gamma[t' + \mathbf{V}\cdot\mathbf{X}'/c^2] \quad \text{(A4b)}$$

In the limit as $\beta \to 0$ $(c \to \infty)$, the Lorentz transformation goes over into the Galilean transformation.

Clarification of Apparent Ambiguities

At the beginning of this paper, a list of five steps was given for the derivation of the Doppler equations for any configuration of transmitters and receivers. It was also mentioned that the improper applications of steps 2 and 3 in the process of deriving a specific Doppler equation may result in an incorrect or only an approximate solution. Step 2 concerns the selection of the coordinate system to which the measurements are to be referenced. Consider configuration A (Fig. 2) in which there is an earth-fixed transmitter and a receiver in the vehicle. The Doppler equations derived contain earth-fixed reference parameters. The vehicle is traveling with a velocity $\mathbf{V}$ with respect to the transmitter. Therefore, from step 2, the earth-fixed transmitter is traveling with a velocity $-\mathbf{V}$ relative to the vehicle.

From step 1, Eq. (A1), the phases are equated:

$$f_t[t - \mathbf{N}\cdot\mathbf{X}/w] = f_r[t' - \mathbf{N}'\cdot\mathbf{X}'/w'] \quad \text{(A5)}$$

where the primed parameters are the measurements in the receiver's (vehicle) reference system. Applying the direct transformation [Eq. (A3)] to the right-hand side of Eq. (A5) yields

$$f_t\{t - \mathbf{N}\cdot\mathbf{X}/w\} = f_r\{\gamma[t - \mathbf{V}\cdot\mathbf{X}/c^2] - \mathbf{N}'\cdot\mathbf{X}/w' - \mathbf{N}'\cdot\mathbf{V}[(\gamma - 1)\mathbf{X}\cdot\mathbf{V}/v^2 - \gamma t]/w'\} \quad \text{(A6)}$$

Equating the coefficients of t gives

$$f_t[1]t = f_r[\gamma + \mathbf{N}'\cdot\mathbf{V}/w']t \quad \text{(A7)}$$

Therefore, the Doppler frequency observed by the receiver on the vehicle, in its own reference system is

$$f_r = f_t(1 - \beta^2)^{1/2}/(1 + \mathbf{N}'\cdot\mathbf{V}/w') \quad \text{(A8)}$$

where $\mathbf{N}'\cdot\mathbf{V} = v\cos\theta'$.

Here we see the interesting ambiguity of two apparently different Doppler equations for the same system, configuration A. Compare Eq. (A8) with (5a). In fact, Eq. (A8) is, in appearance, similar to a different system, configuration B [compare Eq. (A8) with (8a)], because, as stated before, Eqs. (5a) and (8a) contain earth-fixed reference parameters, whereas Eq. (A8) contains vehicle-fixed reference parameters.

Let us now compare the two configuration A equations, (5a) and (A8) under first-order expansions. From Eq. (6) in earth-fixed coordinates

$$f_r \approx f_t[1 - \mathbf{N}\cdot\mathbf{V}/w] \quad \text{(A9a)}$$

From Eq. (A8) in vehicle-fixed coordinates

$$f_r \approx f_t[1 - \mathbf{N}'\cdot\mathbf{V}/w'] \quad \text{(A9b)}$$

Again we encounter the first-order approximation phenomenon that apparently eliminates the necessity for strict adherence to the five steps of derivation. The earth-fixed and vehicle-fixed forms of the approximate Doppler equation, Eqs. (A9a) and (9b), respectively, are identical in appearance.

References

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[4] Moller, C., *The Theory of Relativity* (Oxford University Press, New York, 1960).